ON THE LAGRANGEAN EQUATIONS OF THE HYDRODYNAMICS OF AN INCOMPRESSIBLE VISCOUS FLUID

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In describing the motion of a fluid the Eulerian method is generally used. In the case of an incompressible fluid, which is considered in the present paper, the velocity field $u(\xi, t)$ serves as the Eulerian flow characteristic (the pressure can be expressed by quadratic combinations of the velocities). The Navier-Stokes equations (together with the continuity equation) in principle permit this field to be determined at each moment of time $t > t_0$ for given initial field $u_0(\xi) = u(\xi, t_0)$.

In a series of hydrodynamic problems it becomes necessary to describe the motion of individual (marked) fluid particles or the evolution of surfaces or volumes which consist of fixed fluid particles. In such problems it is more convenient to use the Lagrangean method to describe the fluid motions. This method is of especial interest in the statistical description of turbulent motions.

Thus, Taylor [1] formulated the basic concepts of the theory of turbulent diffusion (which is none other than the statistical effect of the transport of mixtures by moving fluid particles) in terms of the Lagrangean correlation functions of the velocity field. The Lagrangean method holds greater prospects for the subsequent development of the theory of the local structure of turbulence, i.e. the statistical structure of the relative motions in the neighborhood of fixed fluid particles; the use of approximate semi-Lagrangean hydrodynamic equations has already permitted a series of interesting results to be obtained in the study of the spectra of passive mixtures [2,3] and of the turbulent energy spectrum [4] in a region of minimum scales of the turbulent fluctuations, as well as in the statistical description of the stretching of material lines and surfaces in a turbulent flow [5]. Finally, the Lagrangean method has a definite advantage in understanding fluid dynamics as a nonlinear mechanical system whose evolution occurs because of internal interactions among its particles.

Because of the apparent cumbersomeness of the Lagrangean equations of the hydrodynamics of a viscous fluid the structure of these equations has until now not been studied in sufficient measure and they have still not found proper application in specific problems. Also some highly limited information has been successfully obtained [6] on the Lagrangean characteristics of locally-isotropic turbulence without using the dynamic equations.

A report by Pierson [7] was devoted to this problem at the International Symposium on Turbulence in September 1961 in Marseille. However, Pierson did not succeed in revealing the structure of the Lagrangean equations and in writing them in compact form; in addition, the attempt which he carried out to study the linearized Lagrangean equations involved a violation of the continuity condition (which Pierson himself also indicated).

1. Equations for the Cartesian coordinates of the fluid particles. The function $\xi(\chi, t)$ which determines at each moment of time t the coordinates of the fluid particles which are identified by the values of the parameter $\mathbf{x}$ serves as the exhaustive Lagrangean characteristic of the flow of an incompressible fluid. The hydrodynamic equations in principle permit the function $\xi(\mathbf{x}, t)$ to be determined at any $t > t_0$ for given initial values of the velocities of the fluid particles

$$\mathbf{V}_0(\mathbf{x}) = \left[\frac{\partial \boldsymbol{\xi}(\mathbf{x}, t)}{\partial t}\right]_{t=t_0}$$

The connection between the Lagrangean and Eulerian characteristics is given by the relation

$$\partial \boldsymbol{\xi}(\mathbf{x}, t) / \partial t = \mathbf{u}[\boldsymbol{\xi}(\mathbf{x}, t), t] \tag{1.1}$$

The transformation from an Eulerian description to a Lagrangean description leads to the replacement of the independent variables (ξ, t) in the hydrodynamic equations by $(\mathbf{x}, t)$ and to the transformation from the unknown function $\mathbf{u}(\xi, t)$ to the new unknown function $\xi(\mathbf{x}, t)$, which is carried out according to Formula (1.1).

Henceforth, we shall use the initial values of their spatial coordinates as the Lagrangean parameters of the fluid particles, i.e. we shall take

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{x}, t_0) \tag{1.2}$$

Let (ξ^1, ξ^2, ξ^3) and (x^1, x^2, x^3) be the Cartesian components of the vectors ξ and $\mathbf{x}$. Replacing the independent variables (ξ^1, ξ^2, ξ^3, t) by (x^1, x^2, x^3, t) means a transformation from Cartesian coordinates to non-stationary curvilinear and non-orthogonal coordinates which follow the motion of the fluid. Actually, each coordinate of the surface x^i = const consists at all times of the same fluid particles; at the initial moment such surfaces are planes, but with the flow of time they move with the fluid and are distorted. The use of moving coordinates in hydrodynamics was proposed as long ago as 1948 by Zel'manov [8] who called it the method of unitary treatment of the motions of a continuous medium.

In the subsequent calculations we shall use the notation

$$\frac{\partial (A, B, C)}{\partial (x^1, x^2, x^3)} = [A, B, C] \tag{1.3}$$

for the Jacobian with respect to the variables x^1, x^2, x^3.

Without further stipulations we shall make use of the fact that the value of $[A, B, C]$ is not changed for a cyclic permutation and that the sign changes for an acyclic permutation of the variables A, B, C.

In the transformation from the Eulerian variables ξ^α to the Lagrangean variables x^β the infinitesimal transformation matrix

$$T = \left\| \frac{\partial \xi^\alpha}{\partial x^\beta} \right\| \qquad (\det T = |T| = [\xi^1, \xi^2, \xi^3])$$

plays an important role.

According to (1.2) $\partial \xi^\alpha / \partial x^\beta = \delta_{\alpha\beta}$ at the initial moment of time, i.e. the matrix T_0 is represented by a single quantity and $|T_0| = 1$. The quantities $\partial x^\alpha / \partial \xi^\beta$ are the principal elements of the inverse matrix T^{-1}, i.e. the algebraic sums of the elements $\partial \xi^\beta / \partial x^\alpha$ in the matrix T divided by $|T|$. Hence for computing derivatives with respect to the Eulerian variable ξ^i we have

$$\frac{\partial f}{\partial \xi^i} = \frac{1}{|T|} [\xi^j, \xi^k, f] \qquad (ijk) \tag{1.4}$$

Here and subsequently i, j, k are the cyclic permutation of the indices 1, 2, 3. Indeed, having written the left-hand side of Formula (1.4) in the form

$$\frac{\partial f}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \xi^i}$$

(here and subsequently summation is implied by repeated Greek indices) we shall convince ourselves that the same expression is obtained by expanding the determinant on the right-hand side with respect to the elements of the third row.

With the help of Formulas (1.4) and (1.1) we obtain for the divergence of the velocity

$$\frac{\partial u_\alpha}{\partial \xi^\alpha} = \frac{\partial}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial t} = \frac{1}{|T|}([\partial \xi^1/\partial t, \xi^2, \xi^3] + [\xi^1, \partial \xi^2/\partial t, \xi^3] + \\ + [\xi^1, \xi^2, \partial \xi^3/\partial t)]) = \frac{1}{|T|}\frac{\partial |T|}{\partial t}$$

Here u_α are the Cartesian components of the velocity. In the case of an incompressible fluid the divergence of the velocity is identically zero, i.e.

$$\partial |T|/\partial t = 0, \qquad \text{or} \qquad |T| \equiv |T_0| = 1$$

In other words, the continuity equation for an incompressible fluid takes the form

$$[\xi^1, \xi^2, \xi^3] = 1 \tag{1.5}$$

in Lagrangean variables.

We shall make further use of Formula (1.4), letting $|T| = 1$ on the right-hand side. With the help of this formula the expression

$$\triangle_\xi f = \frac{\partial}{\partial \xi^\alpha}\frac{\partial f}{\partial \xi^\alpha} = [\xi^2, \xi^3, [\xi^2, \xi^3, f]] + [\xi^3, \xi^1, [\xi^3, \xi^1, f]] + [\xi^1, \xi^2, [\xi^1, \xi^2, f]] \tag{1.6}$$

is obtained for the Laplacian operator with respect to the Eulerian variables.

The equations of motion of an incompressible viscous fluid in Eulerian variables have the form

$$\frac{du_i}{dt} = -\frac{\partial P}{\partial \xi^i} + \nu \triangle u_i \qquad \left(P = \frac{p}{\rho}\right) \tag{1.7}$$

Here p is the pressure, ρ is the density, and ν is the kinematic coefficient of viscosity. Using Formulas (1.1), (1.4) and (1.6) we transform (1.7) to Lagrangean variables in the following way

$$\frac{\partial^2 \xi^i}{\partial t^2} = -[\xi^j, \xi^k, P] + \nu([\xi^2, \xi^3, [\xi^2, \xi^3, \partial \xi^i/\partial t]] + [\xi^3, \xi^1, [\xi^3, \xi^1, \partial \xi^i/\partial t]] + \\ + [\xi^1, \xi^2, [\xi^1, \xi^2, \partial \xi^i/\partial t]]) \tag{1.8}$$

Equations (1.5) and (1.8) constitute the complete system of equations of the dynamics of an incompressible fluid in Lagrangean variables.

The forces which describe the interaction between the components of a mechanical system correspond to the terms in the equations of motion which are nonlinear with respect to the basic dynamical variables. In the Navier-Stokes equations (1.7) the terms which are nonlinear with respect to the variables u_i are contained in the expression for the acceleration du_i/dt; the forces of inertial interactions between the spatial non-uniformities of the velocity field $\mathbf{u}(\xi, t)$ correspond to them; the pressure gradient as well is expressed through these forces (we emphasize that the viscous forces are described in (1.7) by linear expressions).

The inertial interactions, however, have a relative character – they are eliminated by transformation to a moving observation system. In the Lagrangean equations of motion (1.8) the real forces of interaction between the fluid particles – the pressure gradient and viscous forces – are, on the contrary, described by expressions which are nonlinear with respect to the basic dynamical variables ξ^i.

We note that the viscous interaction forces are described in the Lagrangean equations by nonlinear expressions of fifth degree with respect to the variables ξ^i (whereas in the Navier-Stokes equations the inertial interactions are described by nonlinear expressions of second degree with respect to the variables u_i).

The ratio of the magnitudes of the nonlinear and linear terms in the equations of motion which are characteristic for a given problem can be called the constant of interaction. Thus, in the case of the Navier-Stokes equations the constant of inertial interaction is the ratio of the characteristic magnitudes of the inertial forces and the viscous forces, i.e. the Reynolds number R. In the case of the Lagrangean equations the constant of viscous interaction is the ratio of the characteristic magnitudes of the viscous forces and the total acceleration, i.e. $1/R$. For sufficiently large values of R (characteristic of well-developed turbulence) the inertial interactions which are taken into account in the Eulerian description of the motion are strong; on the contrary, the viscous interactions which are taken into account in the Lagrangean description are weak.

We shall show some special cases in which the form of the Lagrangean equations is somewhat simplified. Let the motion take place only in planes x^3 = const, i.e. $\xi^3 \equiv x^3$ (planar motion). Then using the binomial symbol in square brackets for the two-dimensional Jacobian

$$\frac{\partial (A, B)}{\partial (x^1, x^2)} = [A, B] \tag{1.9}$$

Equations (1.5) and (1.8) can be reduced to the form

$$[\xi^1, \xi^2] = 1$$

$$\frac{\partial^2 \xi^1}{\partial t^2} = -[P, \xi^2] + \nu([\xi^1, [\xi^1, \partial\xi^1/\partial t]] + [\xi^2, [\xi^2, \partial\xi^1/\partial t]] + $$

$$+ [\xi^1, \xi^2, [\xi^1, \xi^2, \partial\xi^1/\partial t]])$$

$$\frac{\partial^2 \xi^2}{\partial t^2} = -[\xi^1, P] + \nu([\xi^1, [\xi^1, \partial\xi^2/\partial t]] + [\xi^2, [\xi^2, \partial\xi^2/\partial t]] +$$

$$+ [\xi^1, \xi^2, [\xi^1, \xi^2, \partial\xi^2/\partial t]]) \tag{1.10}$$

The third equation of motion reduces to the form $[\xi^1, \xi^2, P] = 0$ and indicates that the dependence of P on the four arguments x^1, x^2, x^3, t reduces to dependence on three arguments $\xi^1(\mathbf{x}, t)$, $\xi^2(\mathbf{x}, t)$ and t. In the case of two-dimensional planar motion (when $\xi^3 \equiv x^3$, and ξ^1 and ξ^2 are independent of x^3) the third terms in the parentheses in (1.10) vanish and the third equation of motion takes the form $\partial P/\partial x^3 = 0$.

In the case of plane parallel motion along the x^1- axis (i.e. for $\xi^2 \equiv x^2$, $\xi^3 \equiv x^3$) the continuity equation is equivalent to the formula

$$\xi^1 = x^1 + \int_{t_0}^{t} v(x^2, x^3, t)\, dt \tag{1.11}$$

and the equation of motion acquires the linear form

$$\frac{\partial v}{\partial t} = -\frac{\partial P}{\partial x^1} + \nu\left(\frac{\partial^2 v}{(\partial x^2)^2} + \frac{\partial^2 v}{(\partial x^3)^2}\right) \tag{1.12}$$

The other two equations of motion show that P is a function only of $\xi^1(\mathbf{x}, t)$ and t.

2. The problem of turbulence. To describe statistically the turbulent motion of a fluid we shall consider the field $\boldsymbol{\xi}(\mathbf{x}, t)$ as a random function of space-time points. For a complete statistical description of turbulence one can use a method proposed by Hopf [9] which consists of finding the characteristic functional of the random function $\boldsymbol{\xi}(\mathbf{x}, t)$ defined by the formula

$$\Phi\{\boldsymbol{\eta}(\mathbf{x}, t)\} = \langle e^{i(\boldsymbol{\xi}, \boldsymbol{\eta})} \rangle, \qquad (\boldsymbol{\xi}, \boldsymbol{\eta}) = \int \xi^\alpha(\mathbf{x}, t)\, \eta_\alpha(\mathbf{x}, t)\, d\mathbf{x}dt \tag{2.1}$$

where the integral extends over the whole region of space-time in which the fluid motion occurs and the symbol $\langle A \rangle$ designates the mathematical expectation of the random quantity A.

The functional Φ is an exhaustive statistical characteristic of the random field $\boldsymbol{\xi}(\mathbf{x}, t)$ because for functional arguments of the form

$$\eta(\mathbf{x}, t) = \sum_{k=1}^{n} \mathbf{a}_k \, \delta(\mathbf{x} - \mathbf{x}_k) \, \delta(t - t_k)$$

the values of Φ are the characteristic functions of the probability distributions for values of the field ξ over any finite large number of space-time points $(\mathbf{x}_k, t_k)$.

In addition to the functional Φ, we shall introduce into consideration the operator

$$\Pi\{\eta(\mathbf{x}, t); \mathbf{x}, t\} = \langle P(\mathbf{x}, t) \, e^{i(\xi, \eta)} \rangle \tag{2.2}$$

For the functionals or operators Ψ on a large number of functions $\eta(\mathbf{x}, t)$ we shall introduce the variational derivative operator $D_k(\mathbf{x}, t)$, taking

$$D_k(\mathbf{x}, t) \Psi\{\eta\} = \frac{\delta \Psi\{\eta\}}{\delta \eta_k(\mathbf{x}, t) \, d\mathbf{x} dt} \tag{2.3}$$

Specifically, the following formula for variational differentiation will be needed

$$D_k(\mathbf{x}, t) \, e^{i(\xi, \eta)} = i \xi^k(\mathbf{x}, t) \, e^{i(\xi, \eta)} \tag{2.4}$$

Finally we shall use the identity

$$[A, B, C] = \varepsilon^{\alpha\beta\gamma} \frac{\partial}{\partial x^\alpha} A \frac{\partial}{\partial x^\beta} B \frac{\partial}{\partial x^\gamma} C \tag{2.5}$$

Here $\varepsilon^{\alpha\beta\gamma} = +1$ if (α, β, γ) is a cyclic permutation of the indices $(1, 2, 3)$ and $\varepsilon^{\alpha\beta\gamma} = -1$ if (α, β, γ) is an acyclic permutation; $\varepsilon^{\alpha\beta\gamma} = 0$ if even two of the indices α, β, γ are the same.

Multiplying Equations (1.5) and (1.8) by $e^{i(\xi, \eta)}$ and then applying the mathematical expectation operator, these equations can with the help of Formulas (2.4) to (2.5) be reduced to the form

$$i \varepsilon^{\alpha\beta\gamma} \frac{\partial}{\partial x^\alpha} D_1 \frac{\partial}{\partial x^\beta} D_2 \frac{\partial}{\partial x^\gamma} D_3 \Phi = \Phi \tag{2.6}$$

$$\begin{aligned} \frac{\partial^2}{\partial t^2} D_i \Phi = {} & i \varepsilon^{\alpha\beta\gamma} \frac{\partial}{\partial x^\alpha} D_j \frac{\partial}{\partial x^\beta} D_k \frac{\partial}{\partial x^\gamma} \Pi + \nu \varepsilon^{\alpha\beta\gamma} \varepsilon^{\mu\nu\rho} \frac{\partial}{\partial x^\alpha} \left(D_2 \frac{\partial}{\partial x^\beta} D_3 \frac{\partial^2}{\partial x^\gamma \partial x^\mu} D_2 \frac{\partial}{\partial x^\nu} D_3 + \right. \\ & \left. + D_3 \frac{\partial}{\partial x^\beta} D_1 \frac{\partial^2}{\partial x^\gamma \partial x^\mu} D_3 \frac{\partial}{\partial x^\nu} D_1 + D_1 \frac{\partial}{\partial x^\beta} D_2 \frac{\partial^2}{\partial x^\gamma \partial x^\mu} D_1 \frac{\partial}{\partial x^{\nu_1}} D_2 \right) \frac{\partial^2}{\partial x^\rho \partial t} \cdot D_i \Phi \end{aligned} \tag{2.7}$$

In these equations all operators D_k and Π are taken at the point $(\mathbf{x}, t)$. The operator D_k is linear, therefore Equations (2.6) to (2.7) form a system of linear equations with respect to Φ and Π. Hence, in particular, it follows that $\Pi = L\Phi$ where L is some linear operator. Thus, the problem of the complete statistical description of turbulence reduces to solving linear equations, which is the principal advantage of the method that has been presented (we shall observe, however, that the apparatus for solving the equations in variational derivatives has not yet been devised).

The function $\xi(\mathbf{x}, t)$ is simply determined by the initial velocity field

$$V_0(\mathbf{x}) = \left[\frac{\partial \xi(\mathbf{x}, t)}{\partial t}\right]_{t=t_0}$$

The complete statistical description of this field is given by the characteristic functional

$$V\{\mathbf{y}(\mathbf{x})\} = \left\langle \exp\left[i \int V_{0\alpha}(\mathbf{x})\, y_\alpha(\mathbf{x})\, dx\right]\right\rangle \tag{2.8}$$

Thus, taking

$$\eta_0(\mathbf{x}, t, \tau) = \mathbf{y}(\mathbf{x}) \frac{\delta(t - t_0 - \tau) - \delta(t - t_0)}{\tau} \tag{2.9}$$

it should be required that the functional Φ satisfy the initial condition

$$\lim_{\tau \to 0} \Phi\{\eta_0(\mathbf{x}, t, \tau)\} = V\{\mathbf{y}(\mathbf{x})\} \tag{2.10}$$

in which the functional V is given.

3. The covariant Lagrangean equations and their linearization. The continuity equation and the Navier-Stokes equations can be transformed to Lagrangean variables in such a way that only the contravariant components of the fluid particles defined by the relations

$$v^i = \frac{\partial \xi^\alpha}{\partial t} \frac{\partial x^i}{\partial \xi^\alpha}, \qquad \frac{\partial \xi^i}{\partial t} = v^\alpha \frac{\partial \xi^i}{\partial x^\alpha} \tag{3.1}$$

and the components of the metric tensor of the moving space

$$g_{ik}(\mathbf{x}, t) = \frac{\partial \xi^\alpha}{\partial x^i} \frac{\partial \xi^\alpha}{\partial x^k} \tag{3.2}$$

appear as the unknown functions in them.

We emphasize that the quantities g_{ik} depend on t, i.e. that the metric of the moving space is nonstationary. At the initial moment $t = t_0$ we have $g_{ik} = \delta_{ik}$.

The matrix $G = \| g_{ik} \|$ is the product of the matrices $T^* = \|\partial \xi^k / \partial x^i\|$, $T = \|\partial \xi^i / \partial x^k\|$, whose determinants are the same and in the case of an incompressible fluid are equal to unity; consequently, in this case the determinant $|G|$ of the matrix $\| g_{ik} \|$ is identically equal to unity.

We shall also introduce the inverse matrix $G^{-1} = \| g^{ik} \|$ and the Christoffel symbol of the second kind

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha}\left(\frac{\partial g_{\alpha j}}{\partial x^k} + \frac{\partial g_{\alpha k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha}\right) = \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^j \partial x^k} \tag{3.3}$$

and we shall use the identity

$$\Gamma^\alpha_{\alpha k} = 0$$

which follows from the condition $|G| = 1$.

The expression for the divergence of the velocity can be transformed in the following way

$$\frac{\partial}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial t} = \frac{\partial x^\beta}{\partial \xi^\alpha}\frac{\partial}{\partial x^\beta}\left(v^\gamma \frac{\partial \xi^\alpha}{\partial x^\gamma}\right) = \frac{\partial x^\beta}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\gamma}\frac{\partial v^\gamma}{\partial x^\beta} + v^\gamma \frac{\partial x^\beta}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\beta \partial x^\gamma}$$

The factor of v^γ in the second term is equal to $\Gamma_{\beta\gamma}{}^\beta = 0$, and the first term reduces to the form $\partial v^\beta/\partial x^\beta$. Thus, the continuity equation for an incompressible fluid can be written in the form

$$\frac{\partial v^\beta}{\partial x^\beta} = 0 \tag{3.4}$$

We shall now find the expression for the contravariant component of the acceleration. Using the second formula of (3.1) twice, we obtain

$$w^i = \frac{\partial^2 \xi^\alpha}{\partial t^2}\frac{\partial x^i}{\partial \xi^\alpha} = \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial}{\partial t}\left(v^\beta \frac{\partial \xi^\alpha}{\partial x^\beta}\right) = \frac{\partial v^i}{\partial t} + v^\beta \frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial}{\partial x^\beta}\left(v^\gamma \frac{\partial \xi^\alpha}{\partial x^\gamma}\right) =$$

$$= \frac{\partial v^i}{\partial t} + v^\beta \left(\frac{\partial v^i}{\partial x^\beta} + v^\gamma \Gamma^i_{\beta\gamma}\right)$$

The expression in parentheses in the last formula is the covariant derivative $\nabla_\beta v^i$.

In addition, the contravariant component of the pressure gradient has the form $g^{i\alpha}\partial P/\partial x^\alpha$, and the Laplacian of the velocity is $g^{\alpha\beta}\nabla_\alpha \nabla_\beta v^i$ (we note that the covariant derivatives are permutational because the moving space is Euclidian and, consequently, the corresponding Riemann-Christoffel tensor is equal to zero).

Thus, the equations of motion can be written in the form

$$\frac{\partial v^i}{\partial t} + v^\alpha \nabla_\alpha v^i = -g^{i\alpha}\frac{\partial P}{\partial x^\alpha} + \nu g^{\alpha\beta}\nabla_\alpha \nabla_\beta v^i \tag{3.5}$$

We shall use the Lagrangean equations of hydrodynamics (3.4) to (3.5) to describe small oscillations of the fluid relative to the rest state. We shall consider the quantities v^i to be small and we shall linearize the Equations (3.5), neglecting the terms which are quadratic with respect to v^i and replacing the quantities g^{ik} and the Christoffel symbols with values which correspond to a fluid at rest (i.e. $g^{ik} = \delta_{ik}$ and $\Gamma_{jk}{}^i = 0$).

The linearized equations of motion will have the form

$$\frac{\partial v^i}{\partial t} = -\frac{\partial P}{\partial x^i} + \nu \triangle v^i \tag{3.6}$$

Here ∇ is the Laplace operator with respect to the variables x^α.

These equations together with the continuity equation (3.4) permit the quantities $v^i(\mathbf{x}, t)$ to be determined for given initial values $v^i(\mathbf{x}, t_0) = u_i(\mathbf{x})$ (where u_i are the Cartesian components of the velocity at the initial moment).

According to (3.1) the quantities v^i are expressed nonlinearly in the Cartesian coordinates of the fluid particles ξ^α; however, in using Equations (3.4) and (3.6) there is no need to linearize these expressions, so that the continuity equation remains exact. This is an important advantage of the proposed procedure for linearizing the Lagrangean equations in comparison to the linearization of the equations for the Cartesian coordinates of the fluid particles, which was carried out by Pierson and which is associated with a violation of the continuity equation. We note that after determining the quantities $v^i(\mathbf{x}, t)$ with the help of Equations (3.4) to (3.6) the Cartesian coordinates of the fluid particles $\xi^i(\mathbf{x}, t)$ can be found from the two Equations (3.1) which for known v^i are linear with respect to the quantities ξ^i.

BIBLIOGRAPHY

1. Taylor, G.I., Diffusion by continuous movements. *Proc. Lond. Math.* 20, 196, 1921.

2. Batchelor, G.K., Small-scale variation of convected quantities like temperature in turbulent fluid, Part I. *J. Fluid Mech.* No. 5, p.1, 113, 1959.

3. Batchelor, G.K., Hiwells, I.D. and Townsend, A.A., Small-scale variation of convected quantities like temperature in turbulent fluid, Part II. *J. Fluid Mech.* Vol. 5, p.1, 134, 1959.

4. Novikov, E.A., O spektre energii turbulentnogo potoka neszhimaemoi zhidkosti (On the energy spectrum of the turbulent flow of an incompressible fluid). *Dokl. Akad. Nauk SSSR* Vol. 139, No. 2, 1961.

5. Batchelor, G.K., The effect of homogeneous turbulence on material lines and surfaces. *Proc. Roy. Soc.* A 213, 349, 1952.

6. Monin, A.S., O lagranzhevykh kharakteristikakh turbulentnosti (On the Lagrangean characteristics of turbulence). *Dokl. Akad. Nauk SSSR* Vol. 134, No. 2, 1960.

7. Pierson, W.J., On the transformation of the Navier-Stokes equations into Lagrangean form with selected linear solutions. Rep. at the Intern. Symposium on Turbulence, *JUGG* and *IYTAM* Sept., Marseille, 1961.

8. Zel'manov, A.L., Primenenie soputstvuiushchikh koordinat v nereliativistskoi mekhanike (Application of moving coordinates in nonrelativistic mechanics). *Dokl. Akad. Nauk SSSR* Vol. 11, No. 6, 1948.

9. Hopf, E., Statistical hydromechanics and functional calculus. *J. Rat. Mech. Anal.* 1, 87, 1952.

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